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Magnetization of Type-II Superconductors in the Range of Fields $H_{c1} \le H \le H_{c2}$: Variational Method

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Abstract—A variational method is proposed to find the magnetic field dependence of the magnetization of type-II superconductors in the mixed state by a self-consistent technique. This model allows for suppression of the order parameter to zero at the centers of Abrikosov vortices and also for the magnetic field dependence of the order parameter. The results can be applied to the entire range of fields $H_{c1} \le H \le H_{c2}$ for any values of the Ginzburg–Landau parameter $\kappa > 1/\sqrt{2}$. It is shown that in weak fields where $\kappa \gg 1$ the behavior of the magnetization can be described exactly in the London approximation provided that the correct value of H_{c1} is used. Near the second critical field this dependence shows good agreement with the well-known Abrikosov result. It is also shown that using the concept of isolated vortices and applying the principle of superposition of the fields and currents generated by these vortices to calculate the magnetization gives inaccurate quantitative results even in fairly weak fields. By going beyond these concepts, it was possible to allow more accurately for the influence of the vortex cores on the magnetization behavior in the intermediate range of fields $H_{c1} \ll H \ll H_{c2}$ and to identify the range of validity of various approximations used widely in the literature.

1. INTRODUCTION

The magnetization of type-II superconductors is a fundamental electromagnetic characteristic. It can be used to find various important parameters of the superconductor such as the lower H_{c1} and upper H_{c2} critical fields, and the Ginzburg–Landau parameter κ [1–3]. An enormous number of experimental and theoretical studies have been devoted to magnetization (see, for example, the reviews [4, 5]). In this context it is important to obtain formulas for the magnetization of superconductors which would be suitable for quantitative calculations over a wide range of external magnetic fields. This problem has been discussed in the literature for some time (see, for example, [2–15]). Until recently, however, there was no convenient and reliable approach which could be applied to calculate the magnetization of a type-II superconductor analytically over the entire range of external fields $H_{c1} \le H \le H_{c2}$.

The problem of calculating the magnetic moment M of a superconductor can be solved most easily in weak fields $H \ll H_{c2}$. Here the cores of the Abrikosov vortices occupy only a small part of the volume and $M(H)$ is obtained for $\kappa \gg 1$ using the London approximation where the modulus of the order parameter is assumed to be constant to calculate the local fields and currents outside the core [1–3]. In the London model the dependence of the magnetization M of an ideal isotropic superconductor on the magnetic field H in the range $H_{c1} \ll H \ll H_{c2}$ can be described using the Fetter formula [6]:

$$-4\pi M = H_{c1} - \frac{1}{4\kappa}\{\ln[2\kappa(H - H_{c1})] + 1.34\}. \quad (1)$$

In this formula and subsequently we use a system of units [3] in which all the distances are normalized to the London depth of penetration of the magnetic field λ, the magnetic field is normalized to $H_c\sqrt{2}$ (where H_c is the thermodynamic critical field), the order parameter is normalized to its equilibrium value, and the vector potential is normalized to $\hbar c/2e\xi$, where $\hbar$ is Planck's constant, c is the velocity of light, e is the electron charge, and ξ is the coherence length. The dimensionless values of the local magnetic field, the vector potential, and the order parameter are denoted by $\mathbf{h}$, $\mathbf{a}$, and f. Note that in this system of units the flux quantum is $\Phi_0 = 2\pi/\kappa$ and $H_{c2} = \kappa$. The lower critical field H_{c1} cannot be calculated self-consistently in the London model. For this reason, H_{c1} appears in Eq. (1) as a parameter and for $\kappa \gg 1$ may be written in the form [3]

$$H_{c1} = \frac{1}{2\kappa}(\ln\kappa + \varepsilon). \quad (2)$$

The constant ε is determined by the structure of the order parameter at the vortex core and its value $\varepsilon \approx 0.50$ was determined by Hu [7] by means of a numerical solution of the complete Ginzburg–Landau system of equations (see also [8]). The Fetter dependence (1) differs for $H \longrightarrow H_{c1}$. In the immediate vicinity of H_{c1} the magnetization in the London model can be obtained numerically [14] or analytically using an approximation which only allows for vortex interaction with nearest neighbors in the vortex lattice [3]. In order to extend the validity of the London approximation, various approaches have been developed which make partial allowance for the contribution of the vortex cores to the free energy of the superconductor (see [5, 13]).

The London model cannot be applied in strong fields because the vortex density is high in this case. The behavior of the magnetization near the second critical field is described by the well-known Abrikosov expression [3]:

$$M = \frac{H - H_{c2}}{4\pi\beta_A(2\kappa^2 - 1)}, \quad H_{c2} - H \ll H_{c2}, \qquad (3)$$

where, for a triangular vortex lattice, we have $\beta_A = 1.16$.

In [10, 11] Clem proposed a fairly simple variational model which allows for the structure of the order parameter near the center of the vortex. The following trial function was used for the modulus of the order parameter:

$$f = \frac{f_\infty r}{\sqrt{r^2 + \xi_v^2}}, \qquad (4)$$

where r is the distance from the center of the vortex, ξ_v and f_∞ are variational parameters characterizing the spatial distribution of the order parameter. This model was used to obtain a formula for H_{c1} [10,11] which for $\kappa \gg 1$ may be expressed in the form (2) where $\varepsilon \approx 0.52$ which shows good agreement with the results of [7, 8].

Hao and Clem then generalized this variational model to the case of a regular vortex lattice and obtained a unified formula for $M(H)$ which can be applied over the entire range of fields $H_{c1} \le H \le H_{c2}$ [11]. One of the most important conclusions of this study is that, even in weak fields, the influence of the vortex cores cannot be neglected and consequently the London model cannot generally give an exact result [11,12]. In the range of fields near H_{c2} the dependence $M(H)$ obtained in [11] is almost the same as the Abrikosov result (3). This theory was subsequently generalized to the case of anisotropic superconductors [16]. The model proposed in [11] has been widely used in the literature. The formula for the magnetization has been actively used to analyze experimental data from measurements of the magnetic moment of various superconductors such as: $YBa_2Cu_3O_{7-\delta}$ [17], $YBa_2Cu_4O_8$ [18], $Bi_2Sr_2Ca_2Cu_3O_{10}$ [19], $(Tl,Pb)Sr_2Ca_2Cu_3O_9$ and $Tl_2Ba_2Ca_2Cu_3O_{10}$ [20], $HgBa_2Ca_2Cu_3O_{8+\delta}$ [21, 22], $Hg_{0.8}Pb_{0.2}Ba_{1.5}Sr_2Cu_3O_{8-\delta}$ [23], and $Nd_{1.85}Ce_{0.15}CuO_{4-\delta}$ [24].

In the present paper we show that several errors were made in the derivation of the formula for $M(H)$ in [11]. For example, the expression for the free energy of the vortex lattice F was formulated using the principle of superposition of the fields and currents of isolated vortices, which can only be applied in weak fields. At the same time, also in the calculations of F the transition was made from summation over the reciprocal vortex lattice to integration. The error associated with this transition and also using an inaccurate value of H_{c1} in (1) led these authors [11] to the erroneous conclusion that the London model is incorrectly formulated for $\kappa \gg 1$ even in weak fields. Unfortunately, this statement is now accepted by a whole range of researchers. Additionally, the dependences of the variational parameters ξ_v and f_∞ on the magnetic induction given in [11] (which determine the behavior of the magnetization to a considerable extent) were not obtained self-consistently and do not follow from the expression used for the free energy, but are simply convenient approximations.

Here we use a variational model to obtain a self-consistent derivation of the expression for $M(H)$. In this case, the spatial distribution of the order parameter was simulated using the trial function (4) and the unit cell of the regular vortex lattice was replaced by a circular one (Wigner–Seitz approximation). The formula obtained for $M(H)$ can be applied over the entire range of fields $H_{c1} \le H \le H_{c2}$ for any values of the Ginzburg–Landau parameter $\kappa > 1/\sqrt{2}$. The result for the magnetization in weak fields for $\kappa \gg 1$ agrees with the London dependence (allowing for the exact value of H_{c1}) whereas in strong fields it shows good agreement with the Abrikosov result. The formula obtained for the magnetization can easily be generalized to the case of anisotropic superconductors where the vortices are oriented along one of the principal axes of the crystal. For this orientation a scaling transformation exists which can be used to calculate the magnetization of an anisotropic superconductor from an isotropic one simply by changing the notation of κ [11]. This aspect is considered in Section 2.

We also discuss the correctness of the approximation of isolated vortices in the mixed state of a superconductor. It is shown that even in weak fields, when the density of vortex filaments is still low, using the principle of superposition of the fields created by separate vortices leads to appreciable quantitative errors in calculations of the magnetization.

2. WIGNER–SEITZ APPROXIMATION

In weak fields the distances between the neighboring vortex filaments are many times the dimensions of the vortex cores. This means that the vortices can be considered as independent interacting objects (see, for example, [1–3, 13, 25, 26]). Thus, in weak fields the principle of field superposition is satisfied: the self-induced field of each filament is assumed to be the same as that of an isolated filament and the local field at an arbitrary point in the superconductor is the sum of the fields of all the filaments. The energy of the vortex lattice is expressed as the sum of the self-energies of the filaments and the energies of their pairwise interaction [3, 25]. A particular case of this approach is the London approximation which neglects the influence of the spatial variation of the order parameter in the core of each filament on its field which is valid when $\kappa \gg 1$.

In strong fields the vortex concentration is high and for this reason the concept of independent filament interactions becomes meaningless (see, e.g., [3]). However, as was shown in [27], the local magnetic field in the regular vortex lattice can still be represented as the sum of terms interpreted as contributions from isolated unit cells. In strong fields however, calculation of these contributions is a nontrivial problem. Moreover, this approach is artificial since the vortices are no longer isolated objects and their properties are determined by the lattice as a whole. In this case, it is far simpler to calculate the local magnetic field distribution and the order parameter in an isolated lattice unit cell. The area of the cell is uniquely related to the magnetic induction and the existence of translational invariance in the system yields the boundary condition that the current density at the cell boundary is zero. This method can also be used to obtain the well-known Abrikosov result for the magnetization in fields near H_{c2}. This approach is also convenient for numerical solutions of the Ginzburg–Landau equation over the entire range of external fields $H_{c1} < H < H_{c2}$ [28, 29].

An important simplification in this case is the Wigner–Seitz approximation, i.e., replacing the hexagonal vortex cell with a circle of the same area. In [9] the Wigner–Seitz approximation was applied to find the magnetization in weak fields when $\kappa \gg 1$ and the results show good agreement with the London model. This approximation has frequently been used in numerical calculations of vortex structures [30–34]. It has been found that in Ginzburg–Landau theory [30] and in microscopic superconductivity theory [34] the approximation of a circular cell yields good results not only in weak fields but also near H_{c2}.

In the present paper we propose a variational model to obtain analytic expressions for the magnetization in the Wigner–Seitz approximation. Instead of solving the complete system of Ginzburg–Landau equations, we use the trial function (4) to model the distribution of the order parameter in a Wigner–Seitz cell and the corresponding local magnetic field is calculated from the second Ginzburg–Landau equation. The fact that expression (4) contains two variational parameters means that the vortex shape at the center of the cell can be varied widely for an arbitrary induction.

We shall calculate the magnetic field distribution in a Wigner–Seitz cell. For the case of cylindrical symmetry the second Ginzburg–Landau equation for the magnetic field can be expressed in the form [3]

$$\frac{1}{r}\frac{d}{dr}\left(\frac{r}{f^2}\frac{dh}{dr}\right) = h. \qquad (5)$$

Equation (5) allowing for (4) has the solution

$$h = \alpha K_0\left(f_\infty\sqrt{r^2+\xi_\nu^2}\right) + \beta I_0\left(f_\infty\sqrt{r^2+\xi_\nu^2}\right), \qquad (6)$$

where I_n is an nth-order Bessel function of an imaginary argument, K_n is an nth-order Macdonald function, and α and β are constant coefficients. The values of the constants α and β can be determined from the conditions for quantization of the magnetic field flux through the Wigner–Seitz cell $\Phi = 2\pi/\kappa$ and zero superconducting current $\mathbf{j} = \mathrm{rot}\,\mathbf{h}$ at its interface. This gives:

$$\alpha = \frac{f_\infty}{\kappa\xi_\nu}\frac{I_1(f_\infty\rho)}{K_1(f_\infty\xi_\nu)I_1(f_\infty\rho) - I_1(f_\infty\xi_\nu)K_1(f_\infty\rho)}, \qquad (7)$$

$$\beta = \frac{f_\infty}{\kappa\xi_\nu}\frac{K_1(f_\infty\rho)}{K_1(f_\infty\xi_\nu)I_1(f_\infty\rho) - I_1(f_\infty\xi_\nu)K_1(f_\infty\rho)}, \qquad (8)$$

where we introduce the notation $\rho = \sqrt{R^2+\xi_\nu^2}$, $R = \sqrt{2/B\kappa}$ is the cell radius, $B = 2\pi/\kappa A_{\mathrm{cell}}$ is the magnetic induction, and A_{cell} is the cell area. We stress that this result can be applied for any κ and in particular for $\kappa \sim 1$ when $H_{c1} \sim H_{c2}$ and the concept of independent filaments is only valid in a narrow range of fields near H_{c1}.

The free energy density of the vortex lattice may be expressed in the form

$$F = F_{\mathrm{core}} + F_{\mathrm{em}},$$

where F_{core} is the energy density associated with the change in the order parameter near the centers of the vortices, and F_{em} is the electromagnetic energy density [3, 11]. In Ginzburg–Landau theory F_{core} and F_{em} are given by the expressions [3]

$$F_{\mathrm{core}} = \frac{1}{A_{\mathrm{cell}}}\int\left[\frac{1}{2}(1-f^2)^2 + \frac{1}{\kappa^2}(\mathrm{grad}\,f)^2\right]d^2r, \qquad (9)$$

$$F_{\mathrm{em}} = \frac{1}{A_{\mathrm{cell}}}\int\left[\mathbf{h}^2 + f^2\left(\mathbf{a} + \frac{1}{\kappa}\mathrm{grad}\gamma\right)^2\right]d^2r, \qquad (10)$$

where γ is the phase of the order parameter and integration is performed over the cell area. We shall find the dependence of F on the variational parameters and the magnetic induction. For the electromagnetic energy density using the second Ginzburg–Landau equation from (10) we can easily obtain: $F_{\mathrm{em}} = Bh(0)$, where $h(0)$ is the magnetic field at the center of the vortex filament [11]. If we substitute Eqs. (4) and (6)–(8) into this formula, we have

$$F_{\mathrm{em}} = \frac{Bf_\infty}{\kappa\xi_\nu}\frac{K_0(f_\infty\xi_\nu)I_1(f_\infty\rho) + I_0(f_\infty\xi_\nu)K_1(f_\infty\rho)}{K_1(f_\infty\xi_\nu)I_1(f_\infty\xi_\nu) - I_1(f_\infty\xi_\nu)K_1(f_\infty\rho)}. \qquad (11)$$

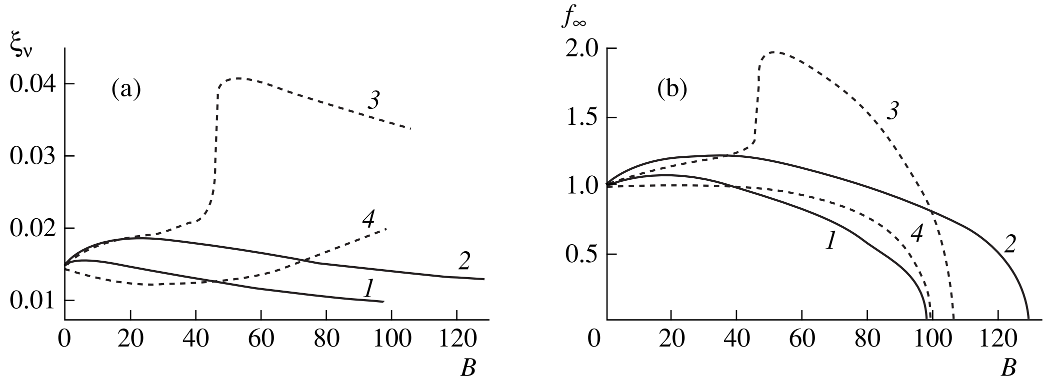

Fig. 1. Curves of $\xi_\nu(B)$ (a) and $f_\infty(B)$ (b) for $\kappa = 100$; (*1*) obtained using the Wigner–Seitz approximation, (*2*) obtained using the isolated vortex approximation, (*3*) using the Hao–Clem continuous approximation, and (*4*) using formulas (24) given in [11].

The expression for F_{core} is derived from Eqs. (4) and (9) and was calculated by Hao and Clem [11]:

$$F_{core} = \frac{1}{2}(1-f_\infty^2)^2 + \frac{1}{2}B\kappa\xi_\nu^2 f_\infty^2\left[(1-f_\infty^2)\ln\left(1+\frac{2}{B\kappa\xi_\nu^2}\right)\right] + \frac{f_\infty^4}{2} - \frac{f_\infty^4}{2+B\kappa\xi_\nu^2} + \frac{Bf_\infty^2(1+B\kappa\xi_\nu^2)}{\kappa(2+B\kappa\xi_\nu^2)^2}. \quad (12)$$

Thus, we have obtained the dependence of the free energy density of the vortex lattice on the magnetic induction and the variational parameters ξ_ν and f_∞. In order to achieve self-consistency in the theory the dependences $\xi_\nu(\kappa, B)$ and $f_\infty(\kappa, B)$ should be obtained by numerically minimizing the function $F(\kappa, B, \xi_\nu, f_\infty)$ with respect to ξ_ν and f_∞. Figures 1a and 1b gives the curves $\xi_\nu(B)$ and $f_\infty(B)$ plotted for the case $\kappa = 100$ (curves *1*). The numerical calculations show that for arbitrary values of κ they can be approximated by the following formulas:

$$\xi_\nu(B,\kappa) = \xi_{\nu 0}\left[\left(1-4.3\left(1.01-\frac{B}{1.05\kappa}\right)^{6.3}\left(\frac{B}{\kappa}\right)^{0.98}\right)\times\left(1-0.56\left(\frac{B}{\kappa}\right)^{0.9}\right)\right]^{1/2}, \quad (13)$$

$$f_\infty(B,\kappa) = \left(1-\frac{B^2}{2.8\kappa^2}\right)\times\left[\left(1+\frac{1.7B}{\kappa}\left(1-\frac{1.4B}{\kappa}\right)^2\right)\left(1-\left(\frac{B}{s\kappa}\right)^4\right)\right]^{1/2}, \quad (14)$$

where the constant is $s = 0.985$ and $\xi_{\nu 0}$ is the value of the parameter ξ_ν at $B = 0$. This value is obtained from the condition $\partial F/\partial\xi_\nu = 0$ for $B = 0$:

$$\kappa\xi_{\nu 0} = \sqrt{2}\left[1-\frac{K_0^2(\xi_{\nu 0})}{K_1^2(\xi_{\nu 0})}\right], \quad (15)$$

from which it follows that $\xi_{\nu 0} \approx \sqrt{2}/\kappa$ for $\kappa \gg 1$.

The magnetic field H is determined from the condition for minimum Gibbs thermodynamic potential $G = F - 2BH$:

$$H = \frac{1}{2}\frac{\partial F}{\partial B}. \quad (16)$$

For the magnetization we then have

$$M = \frac{B-H}{4\pi} = -\frac{1}{8\pi}\frac{\partial}{\partial B}(F-B^2). \quad (17)$$

The magnetization can be conveniently expressed in the form $M = M_{core} + M_{em}$ where the terms

$$M_{core} = -\frac{\partial F_{core}}{\partial B}, \quad M_{em} = -\frac{1}{8\pi}\frac{\partial}{\partial B}(F_{em}-B^2) \quad (18)$$

are the contributions made to the total magnetization by the energy associated with the change in the order parameter at the vortex core and the electromagnetic energy.

For M_{em} using Eq. (11) we then obtain the following expression:

$$-4\pi M_{em} = \frac{f_\infty}{2\kappa\xi_\nu}\times\frac{K_0(f_\infty\xi_\nu)I_1(f_\infty\rho)+I_0(f_\infty\xi_\nu)K_1(f_\infty\rho)}{K_1(f_\infty\xi_\nu)I_1(f_\infty\rho)-I_1(f_\infty\xi_\nu)K_1(f_\infty\rho)} + \frac{1}{2B\kappa^2\xi_\nu^2\rho^2}\times[K_1(f_\infty\xi_\nu)I_1(f_\infty\rho)-I_1(f_\infty\xi_\nu)K_1(f_\infty\rho)]^{-2} - B. \quad (19)$$

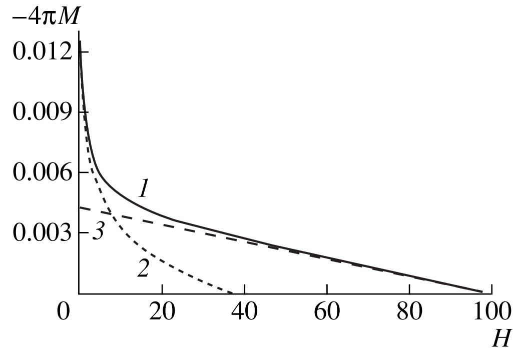

Fig. 2. Curves of $-4\pi M(H)$ over the entire range of fields $H_{c1} \le H \le H_{c2}$ for $\kappa = 100$ obtained using different approximations: (*1*) Wigner–Seitz approximation; (*2*) London approximation [using the Fetter formula (1)] allowing for the exact value of H_{c1}, and (*3*) Abrikosov approximation for strong fields.

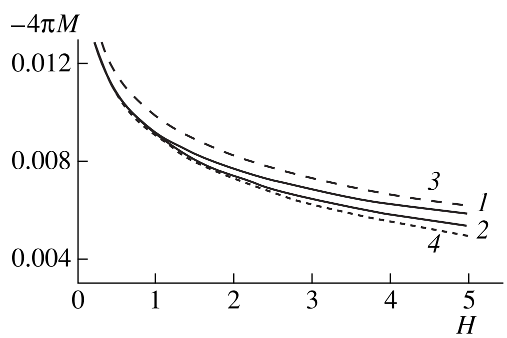

Fig. 3. Curves of $-4\pi M(H)$ in weak fields $H_{c1} \le H \le 0.05H_{c2}$ for $\kappa = 100$ obtained using different approximations: (*1*) Wigner–Seitz approximation, (*2*) isolated vortex approximation, (*3*) Hao–Clem continuous approximation, (*4*) London approximation [using the Fetter formula (1)] allowing for the exact value of H_{c1}.

The relationship for $-4\pi M_{\text{core}}$ from [11] still holds and, in accordance with (9), has the form

$$-4\pi M_{\text{core}} = \frac{\kappa f_\infty^2 \xi_v^2}{2}\left[\frac{1-f_\infty^2}{2}\ln\left(\frac{2}{B\kappa\xi_v^2}+1\right) - \frac{1-f_\infty^2}{2+B\kappa\xi_v^2} + \frac{f_\infty^2}{(2+B\kappa\xi_v^2)^2}\right] + \frac{f_\infty^2(2+3B\kappa\xi_v^2)}{2\kappa(2+B\kappa\xi_v^2)^3}. \quad (20)$$

The dependences $\xi_v(\kappa, B)$ and $f_\infty(\kappa, B)$ are determined by Eqs. (13) and (14). According to Eq. (17), we have $H(B) = B - 4\pi M$. Thus we obtained the implicit dependence $M(H)$.

We shall consider the limiting case $\kappa \gg 1$. Then, in the range of fields $H \ll H_{c2}$ the variational parameters can be assumed to be constant: $\xi_v = \xi_{v0} = \sqrt{2}/\kappa \ll 1$, $f_\infty = 1$ and it also follows from (20) that M_{core} can be assumed to be constant. In this case, Eqs. (19) and (20) can be expanded in powers of ξ_v and we obtain the well-known expression [9]:

$$-4\pi M(B) = H_{c1} + \frac{1}{2\kappa}\left[\frac{K_1(R)}{I_1(R)} + \frac{1}{2I_1^2(R)}\right] - B + O(\xi_v^2). \quad (21)$$

This expression holds as far as H_{c1} where the Fetter dependence (1) diverges. If $H \gg H_{c1}$, we find $R \ll 1$. In this case, the expression for $-4\pi M$ can be expanded not only in powers of ξ_v but also in powers of R and we obtain expression (1) with H_{c1} in the form (2) with $\varepsilon \approx 0.52$.

Figure 2 gives the curve of $-4\pi M(H)$ calculated using Eqs. (19) and (20) for $\kappa = 100$ (curve *1*). Figure 3 gives this curve in weak fields (curve *1*). In weak fields the dependence is the same as the Fetter curve (Fig. 2, curve *2*, Fig. 3, curve *4*). In strong fields it shows good agreement with the Abrikosov result (3) (Fig. 2, curve *3*). Note that the Abrikosov expression fairly accurately describes the behavior of the magnetization as far as fields of around $0.4H_{c2}$. In fields close to H_{c1} where the Fetter formula cannot be applied, our dependence agrees with the calculations [14] for the London model. Thus, in order to calculate the magnetization in weak fields for $\kappa \gg 1$ we can use the London approximation provided that we allow for the correct value of H_{c1}. In fact, in weak fields in the London approximation, the influence of the structure of the order parameter inside the cores of vortex filaments on the self-energy of each filament can be taken into account by introducing the exact value of H_{c1}. At the same time when $\kappa \gg 1$ the structure of the order parameter has a negligible influence on the filament interaction energy because the distances between neighboring filaments are many times greater than their core dimensions. Our result agrees with the conclusion reached by Hao and Clem [11, 12] that the London approximation is inaccurate even in weak fields for $\kappa \gg 1$. The authors of [11] used an inexact value of H_{c1} in the Fetter expression (1) (which is equivalent to using an inexact value of the self-energy of an isolated vortex). In addition, an approximation for the electromagnetic energy was used in [11]. We shall show that the error associated with this approximation is also significant.

Figure 4 gives the curve $M(H)$ for $\kappa = 5$ (curve *1*). Note that in this case the magnetization can only be described using the approximation of independent vortices near H_{c1}. As for large κ, the Abrikosov dependence (3) (Fig. 4, curve *2*) remains valid as far as fields around $0.4H_{c2}$.

The upper critical field in the variational model is defined as the field at which the order parameter in the entire superconductor becomes zero. According to the approximation (14), f_∞ (and thus the order parameter) is zero for $H \approx 0.985\kappa$. This value is fairly close to the true value $H_{c2} = \kappa$. The small difference between H_{c2} and κ can be attributed to the approximate nature of the variational model. As a result, the values of the first and second critical fields calculated using this model cannot be identically equal to the true values H_{c1} and H_{c2}. Thus, although the value of the lower critical field is fairly close to the exact value H_{c1}, it still differs from it. The same applies to the upper critical field.

For practical application of the formula for the magnetization, we can set $s = 1$ in (14). This leads to better agreement between the dependence obtained and the Abrikosov expression (3) for $H \longrightarrow H_{c2}$. In fields below H_{c2} our result remains the same. Small differences from the Abrikosov result for $H \longrightarrow H_{c2}$ can be explained by the fact that the variational model uses a circular vortex cell.

Thus, in the asymptotic limits of weak and strong fields this dependence of the magnetization on the magnetic field agrees with the well-known results: the London dependence (1) (for $\kappa \gg 1$) and the Abrikosov result (3) (in a wide range of values $\kappa > 1/\sqrt{2}$). In this model the values of the first and second critical fields are fairly close to the true values of H_{c1} and H_{c2}. This suggests that this dependence accurately describes the behavior of the magnetization of type-II superconductors in the mixed state.

The magnetic properties of anisotropic superconductors are described by Ginzburg–Landau equations with an effective mass tensor. The variational model can easily be generalized to this case if the vortices are oriented along one of the principal axes of the crystal x_i, $i = 1, 2, 3$. Note that these directions of the external magnetic field are usually used in experimental studies. It was shown in [35] (see also [11]) that in this case, the anisotropic Ginzburg–Landau equations can be transformed to the isotropic form by means of a simple scaling transformation. In order to obtain the dependence of the magnetization on the external field in the anisotropic case from the known dependence for an isotropic superconductor, we need to replace the Ginzburg–Landau parameter κ with $\tilde{\kappa}_\alpha = \kappa u_\alpha^{-1/2}$, where $u_\alpha = m_\alpha/\sqrt{m_1 m_2 m_3}$, and m_i are the effective masses in the direction of the x_i axis (here the vortices are directed along the x_α axis). The case of arbitrary orientation of the vortex filaments relative to the principal axes is studied in [24].

3. APPROXIMATION OF ISOLATED VORTICES

The field h_0 generated by an isolated vortex is a decreasing solution of the Ginzburg–Landau equation (5) over large distances. If the order parameter in the entire superconductor is distributed according to Eq. (4), taking into account the flux quantization condition, we can find the dependence of the field h_0 on the distance from the vortex axis r from (5) [11]:

$$h_0(r) = \frac{f_\infty K_0(f_\infty \sqrt{r^2 + \xi_\nu^2})}{\kappa \xi_\nu K_1(f_\infty \xi_\nu)}. \quad (22)$$

Note that this formula is a particular case of Eq. (6) for $B \longrightarrow 0$.

The trial function (4) was used in [11, 12] to model the distribution of the order parameter in each unit cell of a regular vortex lattice. In these studies the local magnetic field over the entire range of fields $H_{c1} < H < H_{c2}$ was obtained from the sum of the contributions of isolated cells. These contributions should be calculated from the second Ginzburg–Landau equation for a given periodic distribution of the order parameter. Instead it was assumed in [11] that each contribution at an arbitrary point in the superconductor is given by Eq. (22) which is valid for an isolated vortex. In this approximation the local magnetic field $h_l(\mathbf{r})$ may be obtained by the simple superposition:

$$h_l(\mathbf{r}) = \sum_i h_0(|\mathbf{r} - \mathbf{r}_i|). \quad (23)$$

This approach retains the concept of vortices as isolated objects. We shall therefore call it the "isolated vortex approximation."

As in the case of the London approximation, the approach described above should only remain exact in weak fields. Unlike the Wigner–Seitz model, this approximation can be used to study various vortex lattice configurations and also to study the vortex state near the surface. We shall analyze the range of validity of this approach. To do this we shall calculate the superconductor magnetization in this approximation over the entire range of external fields $H_{c1} \le H \le H_{c2}$ and we shall compare this with known results.

As before, the electromagnetic energy density in this case is determined by the formula $F_{em} = Bh(0)$ and the magnetic field at the center of the vortex $h(0)$ is

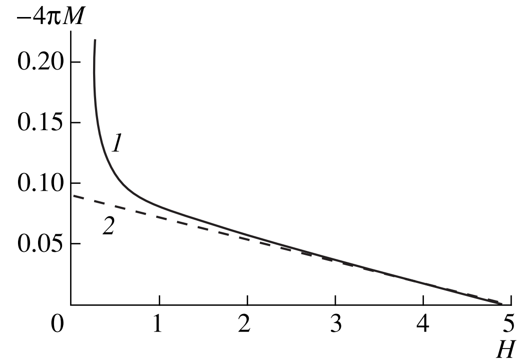

Fig. 4. Curves of $-4\pi M(H)$ in fields $H_{c1} \le H \le H_{c2}$ for $\kappa = 5$ obtained using various approximations: (*1*) Wigner–Seitz approximation; (*2*) Abrikosov result (3).

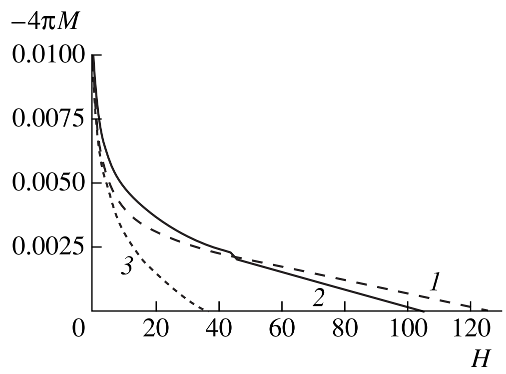

Fig. 5. Curves of $-4\pi M(H)$ in fields $H_{c1} \le H \le H_{c2}$ for κ = 100 calculated using various approximations: (*1*) isolated vortex approximation; (*2*) Hao–Clem continuous approximation, (*3*) London approximation [using the Fetter formula (1)] allowing for the exact value of H_{c1}.

made up of the self-induced field of the vortex and the fields generated by all the other vortices. In this case, for $F_{\rm em}$ for a triangular lattice we have

$$F_{\rm em} = \frac{Bf_\infty}{\kappa\xi_v K_1(f_\infty\xi_v)} \times \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\left[K_0\left(f_\infty\sqrt{d_f^2(3m^2+n^2)+\xi_v^2}\right) + K_0\left(f_\infty\sqrt{d_f^2(m+1/2)^2+d_f^2(n+1/2)^2+\xi_v^2}\right)\right], \quad (24)$$

where $d_f = (4\pi/B\kappa\sqrt{3})^{1/2}$ is the vortex lattice constant. Using Eq. (24), we find the dependence of the magnetization on the magnetic field. The values of the variational parameters are determined by numerical minimization of the free energy density. However, this procedure using Eq. (24) directly is difficult since in strong fields we need to have thousands of terms in each of the single sums to achieve the required accuracy. In the Appendix, Eq. (24) is transformed to a more suitable form for the numerical calculations.

A numerical minimization of the free energy gives the dependences $\xi_v(B)$ and $f_\infty(B)$ plotted in Figs. 1a and 1b for κ = 100 (curves *2*). The corresponding curve $-4\pi M(H)$ is plotted in Fig. 3 (curve *2*) and Fig. 5 (curve *1*) for κ = 100. In this case, the value of the second critical field H_{c2} is higher than the correct value, being approximately $1.29H_{c2}$. Thus, in this approximation the behavior of the magnetization near H_{c2} does not agree with the Abrikosov result because the concept of isolated vortices becomes meaningless here as a result of the substantial overlap of the vortex cores.

For comparison the Fetter curve (*1*) is plotted in Fig. 3 (curve *4*) and Fig. 5 (curve *3*). In weak fields the dependence $M(H)$ in the isolated vortex approximation is almost the same as the corresponding curve in the London approximation (for κ = 100 as far as fields $H \approx 0.02H_{c2}$ which is approximately $80H_{c1}$). At the same time, a comparison of the magnetization curves (Fig. 3, curves *1* and *2*) shows that even in fairly weak fields $H \approx 0.05H_{c2}$ for κ = 100, when the spacings between the vortices are still large, the error associated with using the isolated vortex approximation is quite appreciable and is around 10%. This is because, in accordance with formula (17), the magnetization is determined by the difference between two numbers, each many times greater than the magnetization itself. Thus, even small corrections to F_{em} may be significant.

To conclude this section we note that for $H \ll H_{c2}$ we have calculated the free energy of a square vortex lattice in the isolated vortex approximation. As was to be predicted, this was higher than the free energy of a triangular lattice. Thus, in the London approximation a triangular lattice is thermodynamically more favorable than a square one.

4. CONTINUOUS APPROXIMATION

In [11] a transition was made to summation over the reciprocal lattice to calculate the free energy F_{em}, followed by a transition to the continuous limit, i.e., the sum was approximated by an integral. For $\kappa \gg 1$ the following approximations were given in [11] for $\xi_v(B)$ and $f_\infty(B)$:

$$\xi_v = \xi_{v0}\sqrt{\left(1-2\left(1-\frac{B}{\kappa}\right)^2\frac{B}{\kappa}\right)\left(1+\left(\frac{B}{\kappa}\right)^4\right)}, \qquad f_\infty = \sqrt{1-\left(\frac{B}{\kappa}\right)^4}. \quad (25)$$

It can be seen from (25) that $f_\infty = 0$ for $B = H_{c2} = \kappa$. A numerical check shows that the formulas (25) do not follow from the expression for the free energy density in the continuous approximation [11]. Minimizing F in terms of the parameters ξ_v and f_∞ gives the curves $\xi_v(B)$ and $f_\infty(B)$ plotted in Figs. 1a and 1b for the case κ = 100 (curves *3*). The dependences (25) are also plotted in these figures (curves *4*).

It can be seen from Figs. 1a and 1b that in the field $H_{cr} \approx H_{c2}/2$ the dependences $\xi_v(B)$ and $f_\infty(B)$ exhibit an abrupt jump. This jump leads to a small jump in the magnetization $M(H)$ which is nevertheless incorrect from the physical point of view (Fig. 5, curve *2*). Such an abrupt change in $\xi_v(B)$ and $f_\infty(B)$ occurs because in addition to the absolute minimum of the free energy, a local minimum occurs near the field H_{cr}. As B varies, the absolute and local extrema abruptly change places. It can be seen from Figs. 1b (curve *3*) and Fig. 5 (curve *2*) that the field for which the order parameter in the superconductor becomes zero differs from the true value $H_{c2} = \kappa$. For κ = 100 this difference is approximately 6.6%.

More significantly, using this approximation for F_{em} gives an error in the behavior of $M(H)$ in weak fields.

Thus, the difference between the values of the magnetization obtained using the continuous approximation (Fig. 3, curve *3*) and the self-consistent (Wigner–Seitz) and London approaches (Fig. 3, curves *1* and *4*) is around 10% for $H \approx 0.01H_{c2}$, $\kappa = 100$. It can be seen from Fig. 3 that a similar difference does not occur when the isolated vortex approximation is used systematically (curve *2*). Using the approximations (25) barely alters the behavior of the magnetization in weak fields although it eliminates the jump in $M(H)$ and more accurately describes the behavior of the magnetic moment near H_{c2}.

Consequently, the difference between the magnetization in weak fields in the Hao–Clem model and the magnetization in the London approximation for $\kappa \gg 1$ is a result of the inaccuracy of the approximation [11] for the electromagnetic energy.

5. CONCLUSIONS

Thus, we have used a variational model which allows for the structure of the order parameter inside the vortex core and the dependence of the modulus of the order parameter on the magnetic field to determine the magnetization of a homogeneous isotropic type-II superconductor in the mixed state over the entire range of magnetic fields $H_{c1} \le H \le H_{c2}$. The model has been generalized to the case of anisotropic superconductors when the filaments are oriented along one of the principal axes of the crystal. In weak fields when $\kappa \gg 1$ our results agree with the predictions of the London model, while in fields near the second critical value they agree with the well-known results obtained by Abrikosov. The proposed model is self-consistent and can be applied for a quantitative description of the magnetization of type-II superconductors. We have also analyzed the accuracy of representing the mixed state as a set of isolated vortices for $\kappa \gg 1$. We have shown that calculating the magnetization using these representations yields appreciable quantitative errors in weak fields.

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APPENDIX

Electromagnetic Energy Density of a Vortex Lattice in the Isolated Vortex Approximation

We shall express the field b created by a single vortex array in a convenient form for numerical calculations. We introduce coordinates centered on the axis of one of the vortices with the x-axis perpendicular to the plane in which the vortex array lies, the y-axis lying in this plane and orthogonal to the vortices, and the z-axis directed along the vortex axes. In this case, using (22) we obtain

$$b(x, y) = \frac{f_\infty}{\kappa\xi_v K_1(f_\infty\xi_v)} \times \sum_{m=-\infty}^{\infty} K_0\left(f_\infty\sqrt{(d_f m + y)^2 + x^2 + \xi_v^2}\right). \tag{A.1}$$

Let us perform Fourier transformation of a summand in Eq. (A.1) with respect to the coordinate y:

$$b(x, y) = \frac{f_\infty}{\kappa\xi_v K_1(f_\infty\xi_v)} \sum_{m=-\infty}^{\infty} \int \frac{dq dy}{2\pi} \times \exp(-iqy) K_0\left(f_\infty\sqrt{(d_f m + y)^2 + x^2 + \xi_v^2}\right).$$

Having integrated this expression with respect to y, using known formulas for the definite integrals of the Bessel functions [36] and the relationships

$$\sum_{m=-\infty}^{\infty} \exp(-iqm) = 2\pi \sum_{m=-\infty}^{\infty} \delta(q - 2\pi m),$$

we can obtain

$$b(x, y) = \frac{f_\infty}{\kappa\xi_v K_1(f_\infty\xi_v)} \frac{\pi}{d_f} \sum_{m=-\infty}^{\infty} \left(f_\infty^2 + \frac{4\pi^2 m^2}{d_f^2}\right)^{-1/2} \times \exp\left(-\sqrt{x^2 + \xi_v^2}\sqrt{f_\infty^2 + \frac{4\pi^2 m^2}{d_f^2}}\right) \cos\left(\frac{2\pi m y}{d_f}\right). \tag{A.2}$$

Taking into account Eqs. (24) and (A.2), we have

$$F_{\rm em} = \frac{B\pi}{\kappa\xi_v K_1(f_\infty\xi_v) d_f\sqrt{3}} \times \left\{ 2\sum_{n=1}^{\infty} \exp\left(-f_\infty\sqrt{3d_f^2 n^2 + \xi_v^2}\right) + \exp(-f_\infty\xi_v) + 2\sum_{m=1}^{\infty}\left(1 + \frac{4\pi^2 m^2}{3d_f^2 f_\infty^2}\right)^{-1/2} \exp\left(-\xi_v\sqrt{f_\infty^2 + \frac{4\pi^2 m^2}{3d_f^2}}\right) + 4\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(1 + \frac{4\pi^2 m^2}{3d_f^2 f_\infty^2}\right)^{-1/2} \times \exp\left(-\sqrt{f_\infty^2 + \frac{4\pi^2 m^2}{3d_f^2}}\sqrt{d_f^2 n^2 + \xi_v^2}\right)\right.$$

$$+ 4\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}(-1)^m\left(1+\frac{4\pi^2 m^2}{3d_f^2 f_\infty^2}\right)^{-1/2}$$

$$\times \exp\left(-\sqrt{f_\infty^2+\frac{4\pi^2 m^2}{3d_f^2}}\sqrt{d_f^2(n+1/2)^2+\xi_v^2}\right)$$

$$+ 2\sum_{m=0}^{\infty}\exp(-f_\infty\sqrt{d_f^2(n+1/2)^2+\xi_v^2})\Bigg\}. \qquad (A.3)$$

Although this formula is fairly cumbersome, it is more convenient for numerical calculations than Eq. (24) since only a few terms (several tens) need be taken into account in the double sums on the right-hand side of Eq. (A.3).

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